Australasian Journal of Philosophy
2011; 1–20, i*First* article

MATHEMATICAL EXPLANATIONS OF EMPIRICAL FACTS, AND MATHEMATICAL REALISM

Aidan Lyon

A main thread of the debate over mathematical realism has come down to whether mathematics does explanatory work of its own in some of our best scientific explanations of empirical facts. Realists argue that it does; anti-realists argue that it doesn't. Part of this debate depends on how mathematics might be able to do explanatory work in an explanation. Everyone agrees that it's not enough that there merely be some mathematics in the explanation. Anti-realists claim there is nothing mathematics can do to make an explanation mathematical; realists think something can be done, but they are not clear about what that something is.

I argue that many of the examples of mathematical explanations of empirical facts in the literature can be accounted for in terms of Jackson and Pettit's [1990] notion of program explanation, and that mathematical realists can use the notion of program explanation to support their realism. This is exactly what has happened in a recent thread of the debate over moral realism (in this journal). I explain how the two debates are analogous and how moves that have been made in the moral realism debate can be made in the mathematical realism debate. However, I conclude that one can be a mathematical realist without having to be a moral realist.

1. Introduction

We can break mathematical explanations up into two kinds: (i) mathematical explanations of mathematical facts, and (ii) mathematical explanations of empirical facts.

With respect to the first kind, it is generally thought that a given mathematical fact can be explained by one proof and not by another [Hafner and Mancosu 2005]. There is some controversy over which are the explanatory proofs and which are not [Lange 2009], and the philosophical details of such explanations have not yet been worked out. However, such explanations are not the primary concern of this paper.

The primary concern of this paper is the second kind of mathematical explanation. Some think that there can be mathematical explanations of empirical facts (e.g., Lyon and Colyvan [2008]) and others don't (e.g., Bangu

Australasian Journal of Philosophy
ISSN 0004-8402 print/ISSN 1471-6828 online
http://www.tandf.co.uk/journals DOI: 10.1080/00048402.2011.596216

[2008]). And among those in the first camp, there is some disagreement over actual examples (e.g., Baker [2005] disagrees with Colyvan [2001]). The disagreement—both within the first camp and between the two camps—is over what it takes for an explanation to be genuinely mathematical. Everyone agrees that it's not enough that there merely be some mathematics in the explanation. The second camp thinks there is nothing mathematics can do to make an explanation mathematical. The first camp thinks there can be something, but they are not clear about what that something is (as pointed out by Baker [2009]).

Baker [forthcoming] has made some initial steps towards clearing this issue up. This paper builds on those steps and offers an account of one way in which the mathematics in an explanation of an empirical fact can be doing explanatory work.

The issue of whether mathematics can genuinely be explanatory has an impact on the debate over mathematical realism. Indeed, the literature about mathematical explanation is largely due to the current debate over mathematical realism. After discussing some examples of mathematical explanations of empirical facts, and outlining my account of the explanatory work that the mathematics is doing in these explanations, I'll discuss how this affects the current debate over mathematical realism.

2. Examples

2.1 The Bees

Bees build their honeycombs out of hexagonal cells. Why hexagons, of all shapes?[1]

Darwin [1859: 288] reasoned that natural selection would choose those bees who made their honeycomb in the most efficient manner, minimizing the amount of energy and wax in its construction. This explanation relies on the following mathematical fact (not proven until fairly recently):

> *The Honeycomb Theorem:* A hexagonal grid is the most efficient way to divide a Euclidean plane into regions of equal area with least total perimeter.
>
> Hales [2001: 4]

Lyon and Colyvan [2008] advance Darwin's explanation as an example of a mathematical explanation of an empirical fact. Those bees that minimize the amount of wax they use to build their honeycombs tend to be selected over less efficient bees, and, because of the Honeycomb Theorem, the most efficient way to build a honeycomb is with hexagonal cells. So bees that build their honeycombs out of hexagonal cells have been selected for over evolutionary history.

It is important to note that various other factors play a role in the explanation besides the Honeycomb Theorem. That is, the theorem *by itself*

[1]This example was first discussed by Lyon and Colyvan [2008] and in more detail by Baker [forthcoming].

is no explanation for why the bees use hexagons. We need an assortment of other facts to have an explanation here. For example, the explanation requires the empirical fact that bees that use less wax than other bees are fitter than those other bees. Lyon and Colyvan [ibid.] claim that the package of such facts as a whole—the empirical facts and the mathematical facts—is a mathematical explanation for why bees build their honeycombs out of hexagonal cells.

The following examples are all like this: empirical facts and mathematical facts come together to produce a mathematical explanation of an empirical fact.

2.2 The Cicadas

There are three species of North American cicadas, of the genus *Magicicada*, that have life-cycles of 13 and 17 years. Why these years, of all years?[2]

For these cicadas, it is evolutionarily advantageous to have a life-cycle period that minimizes intersection with the life-cycle periods of other organisms. Prime-numbered life-cycles minimize the intersection with other periods. For example, 12 (a non-prime) intersects with 1, 2, 3, 4, and 6; while 13 (a prime) only intersects with 1. There are various ecological constraints on cicadas that make it advantageous to have a life-cycle between 12 and 18 years. (For example, during the Pleistocene Epoch (about 1.8 million years ago) it was advantageous to have a long life-cycle so as to minimize intersection with unusually cold summers that would occasionally occur during that period.) The only prime life-cycles between 12 and 18 are 13 and 17. So the reason why we see cicadas with life-cycles of 13 and 17 years is that these are the most evolutionary advantageous life-cycles to have, given the ecological constraints and the number-theoretic properties of 13 and 17.

2.3 The Sunflower Seeds

Sunflowers arrange their seeds in an elegant structure of spirals. Why not some other arrangement?

This example is similar to the previous two in that part of the explanation is from evolutionary theory: those sunflowers that pack more seeds into their flowerheads have an evolutionary advantage over those sunflowers that fit fewer. As the flower grows (i.e., as the flowerhead expands) the new seeds are grown at the centre of the flowerhead and older seeds are pushed outwards. After a particular seed has developed, a new one is developed at some angle of rotation from the older one. Given this mechanism of growth, we can consider alternative ways for the sunflower to pack its seeds. Suppose a sunflower produces the seeds with an angle of rotation of, say, 1/7 of 360 degrees. If this is the strategy the sunflower opts for, then as it grows, its

[2] This example was first discussed by Baker [2005], criticised by Bangu [2008] and Daly and Langford [2009], and defended by Baker [2009].

seeds will be packed in seven straight lines, emanating from the centre of the flower head (first image of Figure 1). A similar pattern will emerge if the rotation angle is any rational fraction of 360 degrees. For example, 1/3 of 360 results in the seeds being packed in three straight lines.

Clearly, then, the sunflower cannot choose an angle of rotation that is some rational fraction of 360 degrees, for this will mean that there is always a periodic 'double up', and the space won't be used efficiently. So an irrational rotation angle needs to be chosen. However, not any irrational will do. Suppose the sunflower chooses a rotation angle of 360π mod 360 (which is roughly 50.4). This is an irrational rotational angle; however, since π is well approximated by the rational 22/7, we again see seven arms—though this time they spiral (second image of Figure 1).

It turns out that, in a sense that can be made mathematically precise,[3] some irrationals are more irrational than others. For example, $\surd 2$ is more irrational than π. If the sunflower were to choose a rotation angle of $360\surd 2$ mod 360 (which is roughly 147.6), the sunflower seeds would be more densely packed in the flower head (third image of Figure 1). This configuration, though, is still not optimal. It turns out that the most irrational number is the Golden Ratio, φ, and if the sunflower chooses a rotation angle of the complement of 360φ mod 360 (which is roughly 137.5[4]), the optimal packing of sunflower seeds is achieved (fourth image of Figure 1).

2.4 The Bridges of Königsberg

No one has ever continuously walked over Königsberg's seven famous bridges, passing over each bridge exactly once. Why?[5]

The islands of Königsberg and the bridges that connect them form a connected graph—each island being a vertex and each bridge being an edge

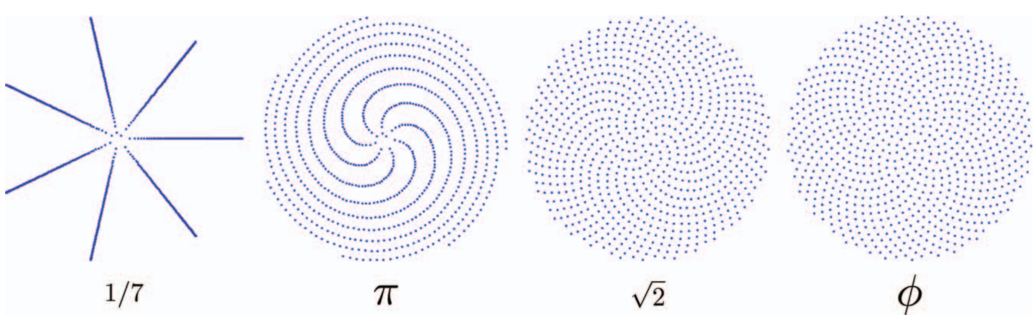

Figure 1. Alternative ways of packing seeds into a sunflower.

[3] Using the theory of continued fractions and rational approximations.

[4] The complement is taken because 360φ mod 360 is roughly 222.5, which is larger than 180. If a rotation angle larger than 180 degrees is chosen, then on each full rotation, only one side of the flower head gets a seed, which is suboptimal.

[5] This example is based on Euler's Königsberg Bridge Problem and is discussed in the context of mathematical explanation by Pincock [2007].

between two vertices. It is a mathematical fact that if a graph has at least one vertex with odd degree (the number of edges that connect to it), then that graph is *non-Eulerian*. A graph is Eulerian if and only if it has a path, beginning and ending at some vertex, that includes each edge of the graph exactly once. There is an island in Königsberg that has three bridges connecting to it, so the graph that the islands and bridges of Königsberg form is non-Eulerian. It is therefore impossible for anyone to continuously walk over Königsberg's seven famous bridges, passing over each bridge exactly once.

2.5 The Kirkwood Gaps

In the asteroid belt between Mars and Jupiter, there are a series of conspicuous looking orbital gaps with very few asteroids in them. These gaps are known as the Kirkwood gaps. Why do they exist?[6]

The gaps exist because of orbital resonances with Jupiter, which produce unstable orbits. Any asteroid in one of these unstable orbits is soon perturbed off the unstable orbit and dragged into a nearby stable one. What's known as an *eigenanalysis* of the mathematical system that represents the physical system delivers the existence and location of the orbital gaps.[7]

2.6 The Plateau Soap Film

Construct a frame for a rectangular box out of some wire. Now dip the wire frame into a soapy solution. When the wire emerges, the soap will form a film within the frame as depicted in Figure 2. Why does the soap form a film of this shape?[8]

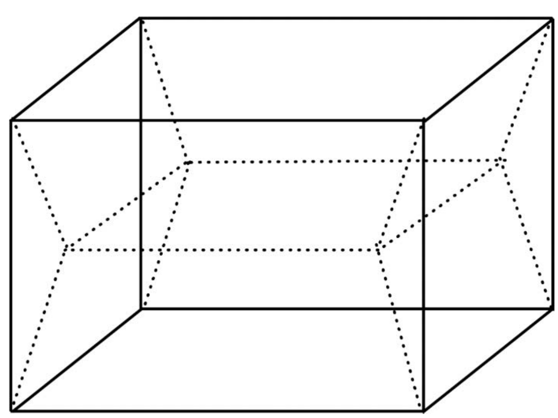

Figure 2. Plateau's Soap Film.

The reason is that the molecules in the soap film arrange themselves in a way that minimizes the potential energy between them, one which

[6]This example was first discussed in the literature by Colyvan [2010].
[7]See Murray and Dermott [2000] for further details.
[8]This example is mentioned by Sober [2011].

corresponds to a way that minimizes the surface area of the film; it is a mathematical fact that the surface bounded by the frame that minimizes surface area is the surface depicted in Figure 2. Plateau originally didn't know what the minimal surface was, but he reasoned that the soap would form a film that would minimize surface area. Conducting this piece of experimental mathematics, Plateau observed the shape of the film, and this guided his mathematical investigation in the theory of minimal surfaces.

3. How Mathematics Can Do Explanatory Work

3.1. Steiner's Proposal

What is it about the above explanations that make them mathematical? Mark Steiner [1978: 18–19] offers a potential answer:

> The distinction between an explanation which uses mathematics and a characteristically mathematical explanation of a physical fact rests upon a proposition which I have argued elsewhere ... that there are mathematical explanations of *mathematical* truth ...
>
> ... The difference between mathematical and physical explanations of *physical* phenomena is now amenable to analysis. In the former, as in the latter, physical and mathematical truths operate. But only in mathematical explanation is this the case: when we remove the physics, we remain with a mathematical explanation—of a mathematical truth!

At first glance, this seems like a plausible proposal; it is endorsed by Lyon and Colyvan [2008: 229]:

> [Hales's] proof explains why a hexagonal grid is the optimal way to divide a surface up into regions of equal area. So the honeycomb conjecture (now the honeycomb theorem), coupled with the evolutionary part of the explanation, explains why the hive-bee divides the honeycomb up into hexagons rather than some other shape ...

However, Baker [forthcoming] has argued that the proposal is false. And it is false for a very simple reason: there are mathematical explanations of empirical facts where the mathematics involved has no (known) mathematical explanation. Baker has argued convincingly, I think, that the example of the bees is a case in point—i.e., that the proof of the Honeycomb Theorem given by Hales [2001] does not *explain* the theorem. Therefore, whatever is doing the explanatory work, it isn't the proof of the theorem. Baker [2009: 623] also considers the proof of the number-theoretic result in the explanation of cicadas' life-cycles not to be explanatory. I'm inclined to agree, and it seems that this is a quite general phenomenon. (However, this is not to say that Steiner's proposal *never* accounts for why an explanation is a mathematical one—see Objection 1 in the next section.)

So the explanatory work of the mathematics in a mathematical explanation is not in the proof of some mathematical fact used in the explanation (at least not generally). We therefore need some other story about what role the mathematics is playing in the above explanations that makes those explanations mathematical. Baker [forthcoming: 18] has argued for a particular constraint on any such story:

> If, as I have argued, mathematical explanations in science are not pure mathematical explanations 'in disguise', then this leaves just a couple of options for how to classify them. Either they are full-fledged scientific explanations, despite their invoking mathematics. Or they are some third category, distinct both from scientific explanations and from pure mathematical explanations. I take it that—on grounds of parsimony, if nothing else—the latter option should only be taken if all other alternatives fail. Hence we ought to treat mathematical explanations in science as scientific explanations.

The account I offer in the next section fits in with the parsimonious option. It understands mathematical explanation of empirical fact as a particular type of scientific explanation: program explanation.

3.2 Program Explanation

In all of the examples from §2, there is an alternative, fine-grained, causal explanation of the explanandum. For example, there is an explanation for why the Kirkwood gaps exist that details the precise locations and movements of the asteroids, their collisions, the exact forces acting on them, etc. None of the above explanations, though, explains at this level of detail. They abstract away from that detail and give a different type of explanation of their explanandum.

The fact that there can be such alternative explanations of the one explanandum is a familiar point in the general philosophy of explanation. Consider two alternative explanations for why a square peg didn't fit through a round hole in a board, where the hole has a diameter equal to the side length of the peg (this example is adapted from Putnam [1975: 295]):

1. Because a particular bit of the peg bumped into a particular bit of the board.
2. Because of the squareness of the peg and the roundness of the hole.

Most agree that the bumping of peg and board bits *caused* the peg not to pass through the hole. And most also seem to agree that the squareness of the peg and the roundness of the hole *explain* why the peg did not pass through the hole, even though these features did not cause this to happen.

Jackson and Pettit [1990] call these two types of explanations *process* and *program* explanations, respectively.[9] Roughly, a process explanation is one that gives a detailed account of the actual causes that led to the event to be

[9]Sterelny [1996] draws a similar distinction, calling the two types of explanation actual sequence and robust process explanations, respectively.

explained. A program explanation, on the other hand, is one that cites a property or entity that, although not causally efficacious, ensures the instantiation of a causally efficacious property or entity that is an actual cause of the explanandum.

To help draw the distinction between the two types of explanation, Jackson and Pettit gave an example of a glass flask that cracks because it had water at boiling temperature in it:

> Although not efficacious itself, the temperature property was such that its realization ensured that there was an efficacious property in the offing: the property, we may presume, involving such and such molecules. The realization of the higher order property did not produce the cracking in the manner of the lower order. But it meant that there would be a suitably efficacious property available, perhaps that involving such and such particular molecules, perhaps one involving others. And so the temperature was causally relevant to the cracking of the glass, under a perfectly relevant sense of relevance, though it was not efficacious. It did not do any work in producing the cracking of the glass—it was perfectly inert—but it had the relevance of ensuring that there would be some property there to exercise the efficacy required.
>
> [Miller 2003: 151–2, after Jackson and Pettit]

The term 'program' comes from an analogy with a computer program [Jackson and Pettit 1990: 114]:

> The analogy is with a computer program which ensures that certain things will happen—things satisfying certain descriptions—though all the work of producing those things goes on at a lower, mechanical level.

In the peg and board example, the squareness of the peg and roundness of the hole program the failure of the peg to pass through the hole, even though all the causal work in producing this event occurs at the level of peg bits bumping into board bits.

One virtue of a program explanation over a process explanation is that the former gives a kind of modal information that the latter lacks [ibid.: 117]. For example, in the case of the flask, someone who is in possession of only the process explanation would not know that had the particular particles that caused the breaking not done so, some other set of particles would have caused the flask to crack. Jackson and Pettit [ibid.: 116] conclude:

> The notion of a programming property does not just explain how an inefficacious property can be relevant to the causation of an event. It also shows how a program explanation can have a significance that remains in the presence of an explanation invoking the corresponding—efficacious property—the corresponding process explanation and more generally in the presence of a lower-order explanation, whether it is of the program or process variety. A program explanation of an event e may provide information which the corresponding process explanation does not supply. Thus, it may be an explanation which the process explanation does not supersede.

In the next section, I'll argue that the explanations from §2 are a particular type of program explanation: ones in which the mathematics is indispensable to the programming. Given the above-mentioned virtue of program explanations, it therefore seems that the explanations from §2 are some of our best scientific explanations.

3.3 Mathematical Explanations as Program Explanations

All of the examples in §2 are program explanations. They cite properties and/or entities which are not causally efficacious but nevertheless program the instantiation of causally efficacious properties and/or entities that causally produce the explanandum. And, importantly, they cite mathematical properties and/or entities that are doing (at least part of) this programming work.

In the example of the soap film, we could know the precise sequence of movements of soap particles that led to their final location without knowing something very important: that pretty much no matter what the precise sequence of movements was, Plateau's soap film would still have formed. This is because the molecules arrange themselves to minimize surface area, and the shape that minimizes surface area is the shape depicted in Figure 2. Changing the exact locations and movements of the soap molecules would not change their final macroscopic form. The mathematical explanation involving the mathematical fact from the theory of minimal surfaces gives us this modal information.

And consider the honeybees. We could explain why honeycombs are hexagonal in terms of various other shapes that the bees tried out in their evolutionary history. For example, the bees were using triangles for a while, then some started using squares, and since this was more efficient, the square bees were fitter and out-reproduced the triangle bees. Eventually some bees hit upon hexagons, which are more efficient than any other shape that has been actually tried by other bees. And so that's why we only see hexagon bees. This would be a process explanation.

However, such an explanation (if it were actually produced) would miss a very important fact: the actual sequence of shapes tried out by the bees is irrelevant to the final outcome. So long as the bees try out hexagons at some point, no matter what other shapes the bees try, the hexagon bees win out. The mathematical explanation for why honeycombs are hexagonal gives us this important modal information.

Similarly, a detailed process explanation could be given for why the *Magicicada* have life-cycles of 13 and 17 years. However, such a detailed historico-ecological explanation would be unsatisfactory in an important respect, as briefly noted by Baker [2009: 621]. The historico-ecological explanation misses the fact that the final evolutionary outcome, the convergence on 13 and 17, is robust with respect to the historico-ecological details. Change those details, and the cicadas would still have the life-cycles that they have.[10] This is an important fact about the evolution of the three

[10]Of course, the life-cycles are not invariant under all changes to the historico-ecological facts. Take away the Pleistocene Epoch, for example, and the cicadas would probably have smaller prime-numbered life-cycles (7 and 11, perhaps). Such background conditions, however, are built into the explanation when given in full detail.

species of *Magicicada*. (For more on the role of program explanations in evolutionary theory see Sterelny [1996].)

Next, the bridges of Königsberg. We could explain why no one has ever continuously walked over Königsberg's seven famous bridges exactly once by citing the details of each walk ever made in Königsberg. However, such an explanation would miss the fact that no matter how anyone chose to go for a walk around Königsberg, they would never pass over each bridge exactly once and end up where they started.

The mathematical explanation of the Kirkwood gaps is also a program explanation. Colyvan [2010: 302] touches on the programming idea in his discussion of the example:

> ... [W]e can seek out a non–mathematical, causal explanation for why each particular asteroid fails to occupy one of the Kirkwood gaps. Each asteroid, however, will have its own complicated, contingent story about the gravitational forces and collisions that that particular asteroid in question has experienced. Such causal explanations are thus piecemeal and do not tell the whole story.

The mathematical explanation is not piecemeal in this way. It gives us modal information about the Kirkwood gaps: e.g., if the complicated contingent story of each asteroid were different, there would still be the Kirkwood gaps in the asteroid belt.

In addition to all of the examples being program explanations, they are also ones in which the mathematics cited is indispensable to the programming of the efficacious properties. For example, if we take away any mention of primeness from the cicada explanation, the explanation falls apart, and there doesn't seem to be anything that would put it back together. Daly and Langford [2009: 657] try to replace primeness with particular durations of previous life-cycles of cicadas and predators and relationships between those durations. But such an explanation quickly loses the modal information that the simpler, mathematical explanation gives: e.g., if some of those durations had been different, the cicadas would still have converged on life-cycles of 13 and 17 years (a version of this point is also made by Baker and Colyvan [manuscript: 6–7] in response to Daly and Langford).

So that's the proposal. An explanation of an empirical fact is mathematical—i.e., it has mathematics doing explanatory work—if the explanation is a program explanation that uses mathematics in a way that is indispensable to the program. Take away the mathematics and the program falls apart. I've shown how the proposal fits all of the examples discussed in this paper (some of which are the primary examples in the literature). It fits many others as well. However, as usual, there are objections.

4. Objections

4.1 Objection 1

This account based on the notion of program explanation doesn't cover any of the following examples of mathematical explanations of empirical facts:

> Why does a moving body contract in the direction of motion? Because of geometric properties of the space-time manifold.
>
> [Colyvan 2001: 50–1]

> Why does light bend when it passes a body of mass? Because bodies of mass curve space-time and light travels along space-time geodesics.
>
> [ibid.: 47–8]

> Why can the displacement of a rigid body about a fixed point always be achieved by rotating the body about a certain angle about a fixed axis? Because rotations are linear transformations of Euclidean space that preserve distances and angles, and such transformations have mathematical properties that guarantee there is an axis of rotation.
>
> [Steiner 1978: 17–18]

As Colyvan [1998: 10] writes of understanding such explanations as program explanations:[11]

> It will work only for those [explanations] in which a fully causal explanation (i.e., one in which all the entities in question are causally efficacious) is on offer as well as the non-causal explanation. Thus, this strategy won't work for the FitzGerald-Lorentz contraction case where only one explanation is on offer and it is non-causal.

The point applies to all three examples: in each example, only one explanation is on offer, it is non-causal, and there are no other, purely causal explanations on offer, so the one explanation on offer cannot be a program explanation.

Response: Agreed; the above explanations are not program explanations. However, I'm not giving a general theory of how mathematics can do explanatory work. My goal is more modest: to carve out a significant set of the examples in the literature and show how the mathematics in them is explanatory. (And in the next section, I'll discuss an interesting impact this has on the larger debate over mathematical realism.) If there are other ways for mathematics to do explanatory work, then that is compatible with what I'm proposing here.

However, one would like to know exactly why the mathematics is explanatory in the above examples.[12] It looks as if a version of Steiner's proposal can account for them, as they appear to be examples where the explanatory work is completely internal to the mathematics.[13] A general account of what it takes for an explanation to be mathematical may be that: either (i) it satisfies Steiner's proposal, or (ii) it satisfies my program

[11]Colyvan is objecting to a different use of program explanations (as a defence of the Eleatic Principle, which I don't endorse). However, his objection is relevant to my thesis too.

[12]It's worth noting, though, that the examples appear not be as clear-cut examples of genuine mathematical explanations as the ones I mentioned earlier—e.g., Baker [2005: 229], a friend of mathematical explanations of empirical facts, is sceptical of them.

[13]Note that an explanation internal to mathematics need not necessarily consist of a proof of a theorem that is explanatory.

explanation proposal. At any rate, I do not need a fully general account for the purposes of this paper.

4.2 Objection 2

The mathematics in all of the purported examples of mathematical explanations of empirical facts is only playing a representational role and so not doing any explanatory work of its own. Daly and Langford [2009] argue for this point with respect to the cicadas, and I grant that a similar argument could be made for the other examples.

Response: If the mathematics is only being used to represent physical properties that are doing the real explanatory work, then it seems the mathematics is not playing a part in the explanation. As Melia [2002: 76] writes:

> ... [W]hen we come to give the geometric explanation of a certain relativistic fact, we may find ourselves indispensably using mathematical objects. But it doesn't follow from this that mathematical objects play a part in the explanation itself, or add to the explanatory power of the theory—for it may be that it is only by using mathematical objects that we are able to pick out a particular geometric property ... [I]t is the geometric properties and the geometric properties alone that do the explaining ...

I am sympathetic to this line of thought. However, the mathematics in each of my examples is not *merely* playing this representational role. It is also playing the programming role, and that's why the mathematics is doing explanatory work. Daly and Langford seem to agree that the mathematics is playing the programming role; however, they appear to be confused by what the programming role is. This leads me to the next objection. (Also, see Baker and Colyvan [manuscript] for detailed criticism of Daly and Langford's arguments.)

4.3 Objection 3

The programming role *is* a purely representational one! Daly and Langford seem to indicate that they think this. Immediately after discussing program explanations and giving some examples, they write [2009: 9]:

> There seems to be a kind of phenomenon endemic in other parts of scientific practice and in everyday explanation-giving which occupies broadly the same role as Melia thinks mathematics plays in scientific explanation, namely, that of indexing [i.e., representing] the properties which are genuinely explanatory.

Response: While a program explanation involving mathematics may use that mathematics to represent some genuinely explanatory properties, it also uses mathematics to do more than that. And that is where the programming comes in. To give a program explanation is to give some modal information about the explanandum. One way to give such modal information is to cite a physical fact—for example, regardless of what Gavrilo Princip was doing, World War I was bound to happen because of the system of armed alliances in Europe at the time. Another way is to cite a mathematical fact—for example, no matter what other shapes the bees tried out, so long as they tried out hexagons, they would fixate on hexagons because that is the most efficient way to partition the 2D plane.

As I mentioned in the introduction, the debate over any purported mathematical explanation of an empirical fact is over whether the mathematics involved plays a *part* in the explanation. Melia [2000] and Daly and Langford [2009] have argued that it is not enough for the mathematics to be *involved* for it to be part of the explanation. It must do something more than be merely involved. I suggest that if the mathematics is *only* playing a representational role, which is one way to be involved, then it is not part of the explanation. However, if it is programming (or if the explanation satisfies Steiner's proposal), then it *is* part of the explanation.

4.4 Objection 4

Program explanations are examples of explanations that cite entities that do no explanatory work. Daly and Langford mention program explanations as explanations that cite entities where those entities do not play a part in the explanation [ibid.: 8–9].

Response: If this is a claim about *all* program explanations, then it is surely false. The temperature of the water in Jackson and Pettit's program explanation is clearly doing explanatory work. Perhaps they mean that *some* program explanations mention entities that do no explanatory work. They refer to Jackson and Pettit [1988] when they mention that program explanations cite entities that do no explanatory work. However, in that paper, Jackson and Pettit use the notion of program explanation to argue that broad content facts can do explanatory work (in explanations of behavioural facts) that is not exhausted by narrow content facts [ibid.: 397]. In a similar fashion, I am using the notion of program explanation to show how mathematical facts can do explanatory work (in explanations of empirical facts) that is not exhausted by the work done by empirical facts.

5. Mathematical Realism

The issue of whether mathematics plays a genuinely explanatory role in our best scientific theories has an impact on a main thread of the debate over mathematical realism. I'll now discuss whether my account of the explanatory role of mathematics in science helps or hurts the realist.

The particular thread of the debate over mathematical realism that is relevant here starts with the Quine-Putnam Indispensability Argument:
(QP1) We ought to be committed to the existence of all and only the entities

that are indispensable to our best scientific theories. (QP2) Mathematical entities are indispensable to our best scientific theories. So, (QP3) We ought to be committed to existence of mathematical entities. There are many objections to this argument; however, my focus here is on an objection due to Melia [2000], who claims that the argument works only if mathematical entities are indispensable to our scientific theories in the *right way*. The idea is that while quantification over mathematical entities may increase the attractiveness or utility of our scientific theories, it doesn't do this in the same way as quantification over physical entities (such as electrons). And so mathematical entities do not earn ontological status the way physical entities such as electrons do.

Colyvan [2002] has responded that mathematical entities do earn their keep, and in the same ways that physical entities do. In particular, Colyvan argues that quantification over mathematical entities contributes indispensably to the *explanatory power* of our scientific theories. In support of his argument, he cites several purported examples of mathematics explaining empirical facts. In a follow-up paper, Melia [2002] objects that Colyvan's examples are not really ones of mathematics doing genuine explanatory work—he argues that they are only cases of mathematics playing a purely representational role.

Baker [2005] continues this thread of the debate by arguing that his example of the cicadas is one in which the mathematics is doing genuine explanatory work. He improves the original Quine-Putnam argument—later calling it an enhanced indispensability argument [Baker 2009: 613]—by replacing (QP1) with: (B1) We ought to be committed to the existence of all and only the entities that are explanatorily indispensable to our best scientific theories. QP2 is also replaced with B2: Mathematical entities are explanatorily indispensable to our best scientific theories. Baker [2009] also responds to the various challenges to his example that have appeared in the literature. One of these objections is again that the mathematics is not doing explanatory work. He concludes that the arguments for this objection do not support it, but admits that he doesn't 'know how to *demonstrate* that the mathematical component [of the explanation] is explanatory' [ibid.: 615, emphasis in original].

It would seem then that the debate has reached an impasse. At this point of the debate, it's not clear who has the more attractive position or where the burden of proof lies. Baker [ibid.: 625] thinks that it lies with the nominalist:

> I think it is reasonable to place the burden of proof here on the nominalist. The way biologists talk and write about the cicada case suggests that they do take the mathematics to be explanatory, and this provides good grounds, at least *prima facie*, for adopting this same point of view.

I'm inclined to agree. However, the mathematical realist may be able to do a bit more than some burden-shifting by using my account of the explanatory role of mathematics.

The notion of program explanation may, then, be usable in the defence of mathematical realism. In §3, I argued that the explanations from §2 are program explanations in which mathematics plays an indispensable role in their programs. As Jackson and Pettit point out, program explanations can give a kind of modal information that alternative process explanations cannot. It seems that the mathematical explanations of §2 are exactly such program explanations, and so they are some of our best scientific explanations. It follows then that (B2) is true: mathematical entities are explanatorily indispensable to our best scientific theories. And so mathematical realism appears to be vindicated. The anti-realist may object that the programming role played by mathematics in explanations is not enough for the mathematics to be playing an explanatory role.[14] However, one would have to maintain that this is true for the program explanations elsewhere in science that involve more mundane entities/properties/quantities (e.g., temperature); or one would have to find a principled reason to distinguish between the two. Both possibilities are not completely implausible, but they're also not very attractive. At any rate, there seems to be a more promising (and interesting) line of attack for the anti-realist.

5.1 Mathematical Realism and Moral Realism

Suppose the anti-realist concedes that the mathematics is explanatorily indispensable to our best scientific theories. Does this require the anti-realist to convert to realism? It appears not. The anti-realist may maintain that although mathematics is indispensable to our best scientific explanations, it still isn't indispensable in the right way—i.e., a way in which ontological status is earned. An analogous move has been made (in this journal) in a very similar and recent thread of the debate over naturalistic, non-reductive moral realism.[15] Mathematical anti-realists, therefore, might be able to take their lead from their moral anti-realist counterparts. In what follows I will briefly summarize this particular thread of the moral realism debate and explain how mathematical anti-realists may use it to their advantage. However, I conclude that the mathematical realist position survives this line of attack.

Nelson [2006] picks up on an argument for naturalistic, non-reductive-moral realism (from hereon I will simply refer to this as 'moral realism') due to Miller [2003] in response to Harman's [1977] challenge to moral realism. The challenge comes from the following argument: (H1) Properties are real iff they figure ineliminably in the best explanation of our experience. (H2) Moral properties do not figure ineliminably in the best explanation of our experience. Therefore, (H3) Moral properties are not real. Although not strictly analogous to the Quine-Putnam argument (in the sense that one can simply replace 'moral properties' with 'mathematical entities' and get the same argument), the similarities are obvious. Indeed, this is effectively the

[14] I suspect this is what Daly and Langford have in mind when they object that program explanations cite entities that do not play a part in those explanations (see Objection 4).
[15] The view that moral properties are natural properties that are explanatorily irreducible to other natural properties.

moral analogue of the challenges to the Quine-Putnam argument put forward by mathematical anti-realists such as Melia [2000] and Field [1980].

Miller's proposal in response to Harman's challenge is for the realist to reject (H2) by arguing that because moral properties play a programming role in some of our best explanations of experience, they figure ineliminably in the best explanation of our experience. Not only does this argument deflate the challenge (if correct), it turns the challenge to moral realism into an argument *for* moral realism. This argument is the moral analogue of the argument I just gave for mathematical realism.

Miller objects to his own argument however, by arguing that (H1) is not quite right; instead, what is true is: (H1′) Properties are real iff they figure ineliminably in the best explanation of experience, considered from the point of view of a subject who suffers no epistemic limitations *vis-à-vis* low-level properties and the process explanations in which they appear.[16] Miller argues that an agent without such epistemic limitations—'God'—'would have no need for program explanations since that agent already knows all of the true counterfactuals specifying the kind of modal information that program explanations can convey' [Miller 2009: 337–8]. In particular, this rules out moral properties' earning any ontological status.

The analogous move for the mathematical realist is to argue that (B1) is not quite true; instead we should replace it with: (B1′) We ought to be committed to the existence of all and only the entities that are explanatorily indispensable to our best scientific theories, considered from the point of view of a subject who suffers no epistemic limitations *vis-á-vis* low-level properties and the process explanations in which they appear.

Nelson responds to Miller for the moral realist by accepting (H1′) and arguing that although God would know that the counterfactuals are true, (s)he wouldn't know *why* they are true. As an example, Nelson claims that God would not know why the following counterfactual about Jackson and Pettit's flask example is true:

> (e*) If molecules x, y, and z had not struck the molecular bonds at place p_n at time t_n with momentum m_n, the glass still would have cracked, due to the striking of other molecules, at other times, or other places, or with slightly different momenta.

He writes [2006: 424–5; emphasis original]:

> ... (e*) requires an explanation, but if God is deprived of ... higher-level property explanation—what could possibly explain (e*)? Deprived of such information, God is left with knowledge of the microphysical state and history of the world right up until the occurrence of *e*, knowledge of all the relevant causal laws, and knowledge of the relevant true counterfactuals (what Miller identifies as 'information about how things would go in relevant possible worlds'). But none of this is sufficient to explain *why* (e*) is true. [Lower-level] facts about the microphysical state and history of the world right up until the

[16]I have taken some liberties with Miller's terminology in order to streamline my discussion of the debate. I don't believe this affects anything.

> occurrence of *e*, and of all the relevant causal laws can't explain it; they can explain only how subsequent history *did in fact* turn out. The true counterfactuals can't explain it, either. They say only how things *do* go in particular possible worlds, but they don't explain *why* it is true that some one of a limited range of possible worlds, out of the limitless panoply of worlds that are possible, will be actualized. Miller stipulates that God, our epistemically unlimited subject, knows that (e*) is true, but it is one thing to know that a proposition is true; it is another thing to have an explanation for it.

Miller [2009] responds to Nelson on this point, but Bloomfield [2009] argues convincingly that Miller's response misses Nelson's point, so I won't dwell on those details here.[17]

The mathematical realist can use this line of argument in response to the analogous challenge by the mathematical anti-realist. (Note that the dialectic between Miller and Nelson has become more much general than the topic of moral realism.) In all of the examples of mathematical explanations of empirical facts that I gave earlier, the explananda were categorical facts: bees have hexagonal-celled honeycombs, cicadas have life-cycles of 13 and 17 years, no one has ever walked over each of Königsberg's seven bridges exactly once, etc. However, for all of these examples, there are also associated counterfactual explananda: e.g., if different people had gone for different walks in Königsberg, no one would have walked over each bridge exactly once. In each example, the mathematical explanation is arguably our best explanation of the relevant counterfactual(s), and, as I have argued, the mathematics is indispensable to those explanations because of the programming roles it plays. So, even if (B1) is replaced with (B1′), it seems that the mathematical realist can respond to the anti-realists' challenge by appealing to the notion of program explanation.

If all of this is right, it seems that these threads of the debates over mathematical realism and moral realism go hand-in-hand: the two realisms stand or fall together. Does a mathematical realist (who is a realist for the reasons given) also have to be a moral realist (for the reasons given above)? So far I have assumed, for the sake of the following the thread of the moral realism debate, that moral properties are ineliminable from some of our best explanations of experience—and that also this is because of their programming role in such explanations. However, the primary example given in support of this premise doesn't seem to be one in which the best explanation must involve moral properties.

The relevant example is of Albert pouring gasoline over a cat and igniting it, causing Jane, who has witnessed this, to form the belief that she has observed something morally wrong. Miller [2003: 154] writes:

> . . . the program explanation provides us with modal information which the corresponding process explanation cannot impart: in possible worlds where the property of being ignited is replaced by other non-moral properties, for example, applying an electric current to the cat, Jane would still have formed

[17]Roughly, Miller argues that God could know the truth of (e*) by examining its truth conditions. However, as Bloomfield points out, that is not the same as knowing why (e*) is true.

> the belief that what Albert did was wrong. Thus, we can mount a program explanation of the formation of Jane's belief in terms of the wrongness of Albert's act, such that the unavailability of that explanation would result in explanatory impoverishment.

However, there seems to be an alternative explanation in terms of Jane's mental states and evolutionary history that is arguably better than the above explanation. Jane forms the belief that what she has observed is wrong because she has the background belief that torturing cats for fun is morally wrong and she believes that she has just seen Albert torture a cat for fun. Her background moral belief can be explained in terms of her evolutionary history (see, e.g., Joyce [2006]). Such an explanation would go something like as follows: Jane is a human and humans have evolved to form moral judgments when they observe certain acts performed by other humans because forming such beliefs was evolutionarily advantageous to human populations in the past (e.g., because they promote successful social behaviour [ibid.: 131]). Such an explanation has all of the virtues that Miller's proposed explanation has. Indeed it has more, for it is more counterfactually robust.[18] Suppose that the moral facts of the example (if there are any) are different, and so Jane's moral beliefs are incorrect. That is, suppose that torturing cats for fun is not morally wrong, but Jane nevertheless believes it is. Jane would still form the belief that what she observed was morally wrong. In the other direction, suppose that the moral facts (if there are any) are the same, but Jane doesn't believe that torturing cats for fun is wrong. Then Jane would not form the belief that what she observed was wrong. So even if there are moral facts in the example, they don't figure ineliminably in the best explanation of Jane's reaction to the situation—it is her *beliefs* that do the explanatory work. The best explanation, therefore, does not involve any moral properties.[19]

It seems, then, that mathematical realists are able to take their lead from the moral realists (of this particular stripe) by using the notion of program explanation to support their mathematical realism, without thereby committing themselves to moral realism in the process.

6. Conclusion

A main thread of the debate over mathematical realism has come down to whether, and how, mathematics can play an explanatory role in our best explanations of empirical facts. By using Jackson and Pettit's notion of program explanation, I have argued that mathematics can do this, and shown *how* it does. My use of the notion of program explanation is analogous to what has happened in the debate over moral realism. Miller suggested that such moral realists can use the notion of program explanation to show how moral properties play an ineliminable explanatory

[18]I owe this point to Darren Bradley.
[19]I lack the space to debunk other examples here, but I believe the strategy generalizes.

role in our best explanations of experience, and thus vindicate moral realism. Nelson defended this argument against Miller's own objection—the objection that to earn ontological status, moral properties should play an explanatory role in our best explanations of experience *from the perspective of a suitably epistemically unlimited agent*. The mathematical anti-realist may try to make a similar move in their debate; however, I have argued that Nelson's response to Miller can be used equally well by the mathematical realist in response to the anti-realist.

All of this suggests that these threads of the debates over (naturalist and non-reductive) moral realism and mathematical realism go hand-in-hand. However, I have also argued that the mathematical realist who accepts my account of the explanatory role of mathematics in science need not be committed to moral realism.[20]

University of Maryland

Received: August 2010
Revised: May 2011

REFERENCES

Baker, A. 2005. Are There Genuine Mathematical Explanations of Physical Phenomena? *Mind* 114/454: 223–38.

Baker, A. 2009. Mathematical Explanation in Science, *The British Journal for the Philosophy of Science* 60/3: 611–33.

Baker, A. forthcoming. Science-Driven Mathematical Explanation, *Mind*.

Baker, A. and M. Colyvan manuscript. Indexing and Mathematical Explanation.

Bangu, S. I. 2008. Inference to the Best Explanation and Mathematical Realism, *Synthese* 160/1: 13–20.

Bloomfield, P. 2009. Moral Realism and Program Explanation: A Very Short Symposium 2: Reply to Miller, *Australasian Journal of Philosophy* 87/2: 343–4.

Colyvan, M. 1998. Can the Eleatic Principle be Justified? *Canadian Journal of Philosophy* 28/3: 313–36.

Colyvan, M. 2001. *The Indispensability of Mathematics*, Oxford: Oxford University Press.

Colyvan, M. 2002. Mathematics and Aesthetic Considerations in Science, *Mind* 111/441: 69–74.

Colyvan, M. 2010. There is No Easy Road to Nominalism, *Mind* 119/474: 285–306.

Daly, C. and Langford, S. 2009. Mathematical Explanation and Indispensability Arguments, *Philosophical Quarterly* 59: 641–58.

Darwin, C. 1859. *On the Origin of Species*, New York: P.F. Collier.

Field, H. 1980. *Science Without Numbers: A Defense of Nominalism*, Princeton: Princeton University Press.

Hafner, J. and P. Mancosu 2005. The Varieties of Mathematical Explanation, in P. Mancosu, K. Jorgensen, and S. Pedersen, eds, *Visualization, Explanation and Reasoning Styles in Mathematics*, Dordrecht: Kluwer: 215–50.

Hales, T. 2001. The Honeycomb Conjecture, *Discrete and Computational Geometry* 25/1: 1–22.

Harman, G. 1977. *The Nature of Morality*, Oxford: Oxford University Press.

Jackson, F. and P. Pettit 1988. Functionalism and Broad Content, *Mind* 97/387: 381–400.

Jackson, F. and P. Pettit 1990. Program Explanation: A General Perspective, *Analysis* 50/2: 107–17.

Joyce, R. 2006. *The Evolution of Morality*, New York: MIT Press.

Lange, M. 2009. Why Proofs by Mathematical Induction Are Generally Not Explanatory. *Analysis* 69/2: 203–11.

Lyon, A. and M. Colyvan 2008. The Explanatory Power of Phase Spaces, *Philosophia Mathematica* 16/2: 227–43.

Melia, J. 2000. Weaseling Away the Indispensability Argument, *Mind* 109/435: 455–80.

Melia, J. 2002. Response to Colyvan, *Mind* 111/411: 75–80.

Miller, A. 2003. *An Introduction to Contemporary Metaethics*, Cambridge: Polity.

Miller, A. 2009. Moral Realism and Program Explanation: A Very Short Symposium 1: Reply to Nelson, *Australasian Journal of Philosophy* 87/2: 337–41.

Murray, C. D. and S. Dermott 2000. *Solar System Dynamics*, Cambridge: Cambridge University Press.

Nelson, M. 2006. Moral Realism and Program Explanation, *Australasian Journal of Philosophy* 84/3: 417–28.

[20]Many thanks to Alan Baker, Darren Bradley, Mark Colyvan, Kenny Easwaran, Daniel Elstein, Fabien Medvecky, and Robbie Williams for helpful discussions and/or comments on earlier drafts. Work on this paper was conducted across four separate institutions: the Sydney Centre for the Foundations of Science at the University of Sydney; the Department of Philosophy at the Australian National University; the Department of Philosophy at the University of Maryland, College Park; and the Tilburg Center for Logic and Philosophy of Science at the University of Tilburg.

Pincock, C. 2007. A Role for Mathematics in the Physical Sciences, *Noûs* 41/2: 253–75.
Putnam, H. 1975. Philosophy and Our Mental Life, in his *Mind, Language and Reality: Philosophical Papers Volume 2*, Cambridge: Cambridge University Press: 291–303.
Sober, E. 2011. Evolution without Naturalism, in *Oxford Studies in Philosophy of Religion, Volume 3*, ed. J. L. Kvanvig, Oxford: Oxford University Press: 187–221.
Steiner, M. 1978. Mathematics, Explanation, and Scientific Knowledge, *Noûs* 12/1: 17–28.
Sterelny, K. 1996. Explanatory Pluralism in Evolutionary Biology, *Biology and Philosophy* 11, 193–214.